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Abstract

We consider *small solutions* of some vibrating mechanical systems with smooth non-linearities for which we provide an approximate solution by using double scale technique; a rigorous proof of convergence of the double scale method is included; for the forced response, a stability result is needed in order to prove convergence in a neighbourhood of a primary resonance.

Double scale expansion of periodic solutions of some vibrating systems, with non linear springs

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1 Introduction

In this work we look for an asymptotic expansion of *small periodic solutions* of free vibrations of a discrete structure without damping and with local non linearity; then the same system with light damping and a *periodic forcing* with frequency close to a frequency of the free system is considered (primary resonance). For a small solution, we recover a behavior with some similarity with the linear case; in particular the amplitude of the forced response reaches a local maximum at the frequency of the free response. On the other hand the frequency of the free response is amplitude dependent and the superposition principle does not apply. The work of Lyapunov [1] is often cited as a basis for the existence of periodic solutions which tends towards linear normal modes as amplitudes tend to zero; the proof of this paper uses the *hypothesis of analycity* of the non linearity involved in the differential system. In [2], we addressed the case of a non linearity which is only lipschitzian and we prove existence of periodic solutions with a constructive proof; in this case the result of Lyapunov obviously may not be applied. Non-linearity of oscillations is a classical theme in theoretical physics, for example at master level, see [3] in Russian or its English or French translation in [4, 5].

Asymptotic expansions have been used for a long time; such methods are introduced in the famous memoir of Poincaré [6]; a general book on asymptotic methods is [7] with french and English translations [8, 9]; introductory material is in [10], [11]; a detailed account of the averaging method with precise proofs

of convergence may be found in [12]; an analysis of several methods including multiple scale expansion may be found in [13]; the case of vibrations with unilateral springs have been presented in [14, 15, 16], [17, 18, 19, 20, 21]; in [22] a numerical approach for large solutions of piecewise linear systems is proposed. A review paper for so called "non linear normal modes" may be found in [23]; it includes numerous papers published by the mechanical community; several application fields have been addressed by the mechanical community; for example in [24] "nonlinear vibro-absorption problem, the cylindrical shell nonlinear dynamics and the vehicle suspension nonlinear dynamics are analyzed".

In the mechanical engineering community the validity of the expansions is assumed to hold; however, this is not straightforward as this kind of expansion is not a standard series expansion and the expansion is usually not valid for all time; for example, this point has been raised in [25]. If the averaging method was carefully analyzed as indicated above, it seems not to be the case for the multiple scale method, the expansion of which is often compared to the one obtained by the averaging method.

Here in a first stage we consider *small solutions* of a system with smooth non-linearities for which we provide an approximate solution by using double scale technique; a rigorous proof of convergence of the double scale method is included; for the forced response, a stability result is needed in order to prove convergence. As an introduction, the next section addresses the one degree of freedom case while the following one considers many degrees of freedom; for free vibrations we find solutions close to a linear normal mode (so called non linear normal modes) and for forced vibrations, we describe the response for forcing frequency close to a free vibration frequency. Preliminary versions of these results may be found in [26] and have been presented in conferences [27, 28]; related results have been presented in [29]. Triple scale expansions is in preparation by N. Ben Brahim [30]. In a forthcoming paper, the non-smooth case will be considered as well as a numerical algorithm based on the fixed point method used in [2]. Such vibrating systems linked to a bar generate acoustic waves; this point will be studied in an other work.

2 One degree of freedom, strong cubic non linearity

In this section, we consider the case of a mass attached to a spring; in the case of a stress-strain law of the form $n = ku + mcu^2 + mu^3$, we find no shift of frequency at first order, so here we concentrate on a stress-strain law with a stronger cubic non linearity:

$$n = ku + mcu^2 + m\frac{d}{\epsilon}u^3,$$

where ϵ is a small parameter which is also involved in the size of the solution as in previous paragraph; the choice of this scaling provides frequencies which are amplitude dependent.

2.1 Free vibration, double scale expansion up to first order

Using second Newton law, free vibrations of a mass attached to such a spring are governed by:

$$\ddot{u} + \omega^2 u + cu^2 + \frac{du^3}{\epsilon} = 0, \tag{1}$$

We look for a *small* solution with a double scale for time; we set

$$T_0 = \omega t, \quad T_1 = \epsilon t, \tag{2}$$

so with $D_0 u = \frac{\partial u}{\partial T_0}, \quad D_1 u = \frac{\partial u}{\partial T_1}$, we obtain

$$\frac{\mathrm{d}u}{\mathrm{d}t} = \omega D_0 u + \epsilon D_1 u, \ \frac{\mathrm{d}^2 u}{\mathrm{d}t^2} = \omega^2 D_0^2 u + 2\epsilon\omega D_0 D_1 u + \epsilon^2 D_1^2 u \tag{3}$$

and we look for a small solution with initial data
$u(0) = \epsilon a_0 + o(\epsilon)$ and $\dot{u}(0) = o(\epsilon)$; we use the *ansatz*

$$u = \epsilon u_1(T_0, T_1) + \epsilon^2 r(T_0, T_1, \epsilon), \tag{4}$$

so we have:

$$\frac{\mathrm{d}u}{\mathrm{d}t} = \epsilon[\omega D_0 u_1 + \epsilon D_1 u_1] + \epsilon^2[\omega D_0 r + \epsilon D_1 r] \tag{5}$$

and

$$\frac{\mathrm{d}^2 u}{\mathrm{d}t^2} = \epsilon\omega^2 D_0^2 u_1 + \epsilon^2[2\omega D_0 D_1 u_1 + \omega^2 D_0^2 r] + \epsilon^3[D_1^2 u_1 + \mathcal{D}_2 r], \tag{6}$$

with

$$\mathcal{D}_2 r = \frac{1}{\epsilon}\left(\frac{\mathrm{d}^2 r}{dt^2} - \omega^2 D_0^2 r\right) = 2\omega D_0 D_1 r + \epsilon D_1^2 r. \tag{7}$$

We plug expansions (4),(6) into (1); by identifying the powers of ϵ in the expansion of equation (1), we obtain:

$$\begin{cases} \omega^2(D_0^2 u_1 + u_1) = 0, \\ (D_0^2 r + r) = \frac{S_2}{\omega^2}, \quad \text{with} \end{cases} \tag{8}$$

$$S_2 = -\frac{1}{\epsilon^2}\left[c(\epsilon u_1 + \epsilon^2 r)^2 + \frac{d}{\epsilon}(\epsilon u_1 + \epsilon^2 r)^3\right] - 2\omega D_0 D_1 u_1 - \epsilon\mathcal{R}(u_1, r, \epsilon), \tag{9}$$

with

$$\mathcal{R} = D_1^2 u_1 + \mathcal{D}_2 r; \tag{10}$$

we can manipulate to obtain:

$$S_2 = -\left[cu_1^2 + du_1^3 + 2\omega D_0 D_1 u_1 + \epsilon R(u_1, r, \epsilon)\right], \tag{11}$$

where

$$R(u_1, r, \epsilon) = \left[\mathcal{R} + 2cu_1 r + 3du_1^2 r + \epsilon\rho(u_1, r, \epsilon)\right], \tag{12}$$

with a polynomial $\rho(u_1, r, \epsilon) = cr^2 + 3du_1r^2 + \epsilon dr^3$.

We set $\theta(T_0, T_1) = T_0 + \beta(T_1)$ noticing $D_0\theta = 1,\ D_1\theta = D_1\beta$; we solve equation (8) with:

$$u_1 = a(T_1)\cos(\theta) \tag{13}$$

and we obtain

$$S_2 = \frac{-ca^2}{2}(1+\cos(2\theta)) - \frac{da^3}{4}\left(\cos(3\theta) + 3\cos(\theta)\right) + \\ 2\omega(D_1 a\sin(\theta) + aD_1\beta\cos(\theta)) - \epsilon R(u_1, r, \epsilon); \tag{14}$$

we gather terms at angular frequency 1:

$$S_2 = -\frac{da^3}{4}3\cos(\theta) + 2\omega\left[D_1 a\sin(\theta) + aD_1\beta\cos(\theta)\right] + S_2^\sharp - \epsilon R(u_1, r, \epsilon), \tag{15}$$

where

$$S_2^\sharp = \frac{-ca^2}{2}(1+\cos(2\theta)) - \frac{da^3}{4}\cos(3\theta). \tag{16}$$

By imposing

$$D_1 a = 0 \text{ and } \quad 2\omega a D_1\beta = 3\frac{da^3}{4}, \text{ so that}$$
$$a = a_0, \quad \beta = \beta_0 T_1 \text{ with } \beta_0 = 3\frac{da^2}{8\omega}T_1, \tag{17}$$

we have canceled the constant involved in β_0 with the choice of zero initial condition and we get that $S_2 = S_2^\sharp - \epsilon R(u_1, r, \epsilon)$ no longer contains any term at frequency 1.

In order to show that r is bounded, after eliminating terms at angular frequency 1, we go back to the t variable in the second equation (8).

$$\ddot{r} + \omega^2 r = \frac{\tilde{S}_2}{\omega^2}, \quad \text{with} \tag{18}$$
$$\tilde{S}_2 = S_2^\sharp(t,\epsilon) - \epsilon\tilde{R}(u_1, r, \epsilon), \text{ where} \tag{19}$$
$$S_2^\sharp(t,\epsilon) = \frac{-ca^2}{2}\left[1+\cos(2(\omega t + \beta(\epsilon t)))\right] - \frac{da^3}{4}\cos(3(\omega t + \beta(\epsilon t))) \tag{20}$$
$$= \frac{-ca^2}{2}(1+\cos(2(\omega t + \beta_0\epsilon t)) - \frac{da^3}{4}\left(\cos(3(\omega t + \beta_0\epsilon t))\right), \tag{21}$$
$$\text{with } \tilde{R}(u_1, r, \epsilon) = R(u_1, r, \epsilon) - \mathcal{D}_2 r, \tag{22}$$

in which the remainder $\tilde{R}$ is expressed with variable t.

Proposition 2.1. *There exists $\gamma > 0$ such that for all $t \le t_\epsilon = \frac{\gamma}{\epsilon}$, the solution of (1), with $u(0) = \epsilon a_0 + o(\epsilon),\ \dot{u}(0) = o(\epsilon)$, satisfies the following expansion*

$$u(t) = \epsilon a_0\ \cos(\nu_\epsilon t)\ +\ \epsilon^2 r(\epsilon, t),$$

where

$$\nu_\epsilon = \omega + 3\epsilon \frac{da^2}{8\omega} \tag{23}$$

and r is uniformly bounded in $C^2(0, t_\epsilon)$.

Proof. Let us use lemma 5.1 with equation (18); set $S = S_2^\sharp$; as we have enforced (17), it is a periodic bounded function orthogonal to $e^{\pm it}$, it satisfies lemma hypothesis; similarly set $g = \tilde{R}$; it is a polynomial in variable r with coefficients which are bounded functions, so it is a lipschitzian function on bounded subsets and satisfies lemma hypothesis. □

2.2 Forced vibration, double scale expansion of order 1

2.2.1 Derivation of the expansion

Here we consider a similar system with a sinusoidal forcing at a frequency close to the free frequency (so called primary resonance); in the linear case, without damping, it is well known that the solution is no longer bounded when the forcing frequency goes to the free frequency. Here, we consider the mechanical system of previous section but with periodic forcing and we include some light damping term; the scaling of the forcing term is chosen so that the expansion works properly; this is a known difficulty, for example see [31].

$$\ddot{u} + \omega^2 u + \epsilon\lambda\dot{u} + cu^2 + \frac{du^3}{\epsilon} = \epsilon^2 F \cos(\tilde{\omega}_\epsilon t). \tag{24}$$

We assume positive damping, $\lambda > 0$ and excitation frequency $\tilde{\omega}_\epsilon$ is *close* to an eigenfrequency of the linear system in the following way:

$$\tilde{\omega}_\epsilon = \omega + \epsilon\sigma. \tag{25}$$

We look for a small solution with a double scale expansion; to simplify the computations, the fast scale T_0 is chosen ϵ dependent and we set:

$$T_0 = \tilde{\omega}_\epsilon t, \quad T_1 = \epsilon t \text{ and } \quad D_0 u = \frac{\partial u}{\partial T_0},\ D_1 u = \frac{\partial u}{\partial T_1}, \tag{26}$$

so

$$\frac{\mathrm{d}u}{\mathrm{d}t} = \tilde{\omega} D_0 u + \epsilon D_1 u \text{ and } \frac{\mathrm{d}^2 u}{\mathrm{d}t^2} = \tilde{\omega}_\epsilon^2 D_0^2 u + 2\epsilon\tilde{\omega}_\epsilon D_0 D_1 u + \epsilon^2 D_1^2 u; \tag{27}$$

equation (25) provides

$$\tilde{\omega}_\epsilon^2 = \omega^2 + 2\epsilon\omega\sigma + \epsilon^2\sigma^2. \tag{28}$$

With (25, (26), (27), (28) and the *ansatz*

$$u = \epsilon u_1(T_0, T_1) + \epsilon^2 r(T_0, T_1, \epsilon), \tag{29}$$

we obtain:

$$\frac{\mathrm{d}u}{\mathrm{d}t} = \epsilon\frac{du_1}{\mathrm{d}t} + \epsilon^2\frac{\mathrm{d}r}{\mathrm{d}t} = \epsilon\frac{du_1}{\mathrm{d}t} + \epsilon^2\omega D_0 r + \epsilon^2(\frac{\mathrm{d}r}{\mathrm{d}t} - \omega D_0 r) = \tag{30}$$

$$\epsilon[\tilde{\omega}D_0 u_1 + \epsilon D_1 u_1] + \epsilon^2\omega D_0 r + \epsilon^2(\frac{\mathrm{d}r}{\mathrm{d}t} - \omega D_0 r) = \tag{31}$$

$$\epsilon[\omega D_0 u_1 + \epsilon\sigma D_0 u_1 + \epsilon D_1 u_1] + \epsilon^2\omega D_0 r + \epsilon^2(\frac{\mathrm{d}r}{\mathrm{d}t} - \omega D_0 r), \tag{32}$$

where we remark that $\frac{\mathrm{d}r}{\mathrm{d}t} - \omega D_0 r = \epsilon\sigma D_0 r + \epsilon D_1 r$ is of degree 1 with respect to ϵ. For the second derivative, as for the case without forcing, we introduce

$$\mathcal{D}_2 r = \frac{1}{\epsilon}\left(\frac{\mathrm{d}^2 r}{\mathrm{d}t^2} - \omega^2 D_0^2 r\right), \text{ with the expansion} \tag{33}$$

$$\mathcal{D}_2 r = 2\omega[\sigma D_0^2 r + D_0 D_1 r] + \epsilon\left[\sigma^2 D_0^2 r + 2\sigma D_0 D_1 r + D_1^2 r\right], \tag{34}$$

$$\frac{\mathrm{d}^2 u}{\mathrm{d}t^2} = \epsilon\frac{\mathrm{d}^2 u_1}{\mathrm{d}t^2} + \epsilon^2\frac{\mathrm{d}^2 r}{\mathrm{d}t^2} = \epsilon\frac{\mathrm{d}^2 u_1}{\mathrm{d}t^2} + \epsilon^2\omega^2 D_0^2 r + \epsilon^3\mathcal{D}_2 r \tag{35}$$

$$= \epsilon\left[\tilde{\omega}^2 D_0^2 u_1 + 2\epsilon\tilde{\omega} D_0 D_1 u_1 + \epsilon^2 D_1^2 u_1\right] \tag{36}$$

$$+ \epsilon^2\omega^2 D_0^2 r + \epsilon^3\mathcal{D}_2 r \tag{37}$$

$$= \epsilon\big\{\omega^2 D_0^2 u_1 + 2\epsilon\omega\left(\sigma D_0^2 u_1 + D_0 D_1 u_1\right) + \tag{38}$$

$$\epsilon^2\left[\sigma^2 D_0^2 u_1 + 2\sigma D_0 D_1 u_1 + D_1^2 u_1\right]\big\} \tag{39}$$

$$+ \epsilon^2\omega^2 D_0^2 r + \epsilon^3\mathcal{D}_2 r; \tag{40}$$

the last term in the right hand side will be part of the remainder R of equation (42). We plug previous expansions into (24); we obtain:

$$\begin{cases} \omega^2(D_0^2 u_1 + u_1) = 0, \\ D_0^2 r + r = \frac{S_2}{\omega^2}, \quad \text{with} \end{cases} \tag{41}$$

$$S_2 = -\left\{cu_1^2 + du_1^3 + 2\omega[D_0 D_1 u_1 + \sigma D_0^2 u_1] + \lambda\omega D_0 u_1\right\} \tag{42}$$

$$+F\cos(T_0) - \epsilon R(u_1, r, \epsilon) \tag{43}$$

and with

$$R(u_1, r, \epsilon) = D_1^2 u_1 + 2cu_1 r + 3du_1^2 r + \sigma^2 D_0^2 u_1 + 2\sigma D_0 D_1 u_1 + \tag{44}$$

$$\lambda(\omega D_0 r + \sigma D_0 u_1 + D_1 u_1) + \mathcal{D}_2 r \tag{45}$$

$$+\lambda(\frac{\mathrm{d}r}{\mathrm{d}t} - \omega D_0 r) + \epsilon\rho(u_1, r, \epsilon). \tag{46}$$

Set $\theta(T_0, T_1) = T_0 + \beta(T_1)$. We solve the first equation of (41) :

$$u_1 = a(T_1)\cos(\theta) \tag{47}$$

then we use $T_0 = \theta(T_0, T_1) - \beta(T_1)$, and we obtain

$$S_2 = \frac{-ca^2}{2}(1+\cos(2\theta)) - \frac{da^3}{4}(\cos(3\theta) + 3\cos(\theta)) + 2\omega(D_1 a \sin(\theta) + aD_1\beta\cos(\theta)) + 2\sigma\omega a\cos(\theta) + a\lambda\omega\sin(\theta) + F\sin(\theta)\sin(\beta(T_1)) + F\cos(\theta)\cos(\beta(T_1)) - \epsilon R(u_1, r, \epsilon), \quad (48)$$

or

$$S_2 = [2\omega D_1 a + \lambda a\omega + F\sin(\beta)]\sin(\theta) + \left[2\omega a D_1\beta + 2\sigma\omega a - \frac{3da^3}{4} + F\cos(\beta)\right]\cos(\theta) + S_2^\sharp - \epsilon R(u_1, r, \epsilon), \quad (49)$$

with

$$S_2^\sharp = \frac{-ca^2}{2}(1+\cos(2\theta)) - \frac{da^3}{4}(\cos(3\theta))\,; \quad (50)$$

note that $S_2^\sharp$ is a periodic function with frequency strictly multiple of 1.

Orientation. By enforcing

$$\left\{\begin{array}{l} 2\omega D_1 a + \lambda a\omega = -F\sin(\beta), \\ 2a\omega D_1\beta + 2\sigma\omega a - \frac{3da^3}{4} = -F\cos(\beta), \end{array}\right. \quad (51)$$

$S_2 = S_2^\sharp - \epsilon R(u_1, r, \epsilon)$ contains neither term at frequency 1 nor at a frequency which goes to 1; this point will enable to justify this expansion under some conditions; first, we study stationary solution of this system and the stability of the dynamic solution in a neighborhood of the stationary solution.

2.2.2 Stationary solution and stability

Let us consider the stationary solution of (51), it satisfies:

$$\left\{\begin{array}{l} \frac{1}{2\omega}[\lambda a\omega + F\sin(\beta)] = 0, \\ \frac{1}{2\omega}\left[\left(2\omega\sigma - \frac{3da^2}{4}\right) + \frac{F\cos(\beta)}{a}\right] = 0, \end{array}\right. \quad (52)$$

Now, we study the stability of the solution of (51), in a neighborhood of this stationary solution noted $(\bar{a}, \bar{\beta})$; set $a = \bar{a} + \tilde{a}, \beta = \bar{\beta} + \tilde{\beta}$, the linearized system is written

$$\begin{pmatrix} D_1\tilde{a} \\ D_1\tilde{\beta} \end{pmatrix} = J\begin{pmatrix} \tilde{a} \\ \tilde{\beta} \end{pmatrix};$$

manipulating, we obtain the jacobian matrix.

$$J = \begin{pmatrix} -\frac{\lambda}{2} & -\frac{F}{2\omega}\cos(\bar{\beta}) \\ \frac{9d\bar{a}}{8\omega} - \frac{\sigma}{\bar{a}} & \frac{F}{2\omega\bar{a}}\sin(\bar{\beta}) \end{pmatrix} = \begin{pmatrix} -\frac{\lambda}{2} & a(\sigma - \frac{3d\bar{a}^2}{8\omega}) \\ \frac{9d\bar{a}}{8\omega} - \frac{\sigma}{\bar{a}} & -\frac{\lambda}{2} \end{pmatrix}. \quad (53)$$

The matrix trace is $-\lambda$, and the determinant is

$$det(J) = \frac{\lambda^2}{4} - (\frac{9d\bar{a}^2}{8\omega} - \sigma)(\sigma - \frac{3d\bar{a}^2}{8\omega});$$

we notice that the determinant is strictly positive for $\sigma = 0$ so by continuity, it remains positive for σ small; moreover $\frac{d}{d\sigma}det(J) < 0$ for $\sigma < 0$ so $det(J) > 0$ for $\sigma < 0$; by studying the trinomial in σ, we notice that the determinant is positive when this semi-implicit inequality is satisfied: $\sigma \leq \frac{3d\bar{a}^2}{4\omega} - \frac{1}{2}\sqrt{\frac{9d^2\bar{a}^4}{16\omega^2} - \lambda^2}$; so in these conditions, the two eigenvalues are negative; then the solution of the linearized system goes to zero; with the theorem of Poincaré-Lyapunov (look in the appendix for the theorem 5.1,) when the initial data is close enough to the stationary solution, the solution of the system (51), goes to the stationary solution. We expand this point, set

$$y = \begin{pmatrix} a \\ \beta \end{pmatrix} \quad G(y) = \begin{pmatrix} -\lambda a\omega & -F\sin(\beta) \\ -\left(2\omega\sigma - \frac{3da^2}{4}\right) & -\frac{F\cos(\beta)}{a} \end{pmatrix}; \tag{54}$$

the system (52) may be written $\dot{y} = G(y)$; denote $\bar{y} = \binom{\bar{a}}{\bar{\beta}}$, the solution of (52); perform the change of variable $y = \bar{y} + x$, we have $G(\bar{y} + x) = G(\bar{y}) + Jx + g(x)$, with $g(x) = o(\|x\|)$; the theorem 5.1 may be applied with $A = J,\ B = 0$, here the function g does not depends on time.

Proposition 2.2. *If $\sigma \leq \frac{3d\bar{a}^2}{4\omega} - \frac{1}{2}\sqrt{\frac{9d^2\bar{a}^4}{16\omega^2} - \lambda^2}$, the stationary solution of (51) is stable in the sense of Lyapunov (if the dynamic solution starts close to the stationary solution of (52), it remains close to it and converges to it); to the stationary case corresponds the approximate solution of (24) $u_1 = \bar{a}\cos(T_0 + \bar{\beta})$, it is periodic; for an initial data close enough to this stationary solution, $u_1 = a(T_1)\cos(T_0 + \beta(T_1))$, with a, β solutions of (51); it goes to the solution (52) $\bar{a}, \bar{\beta}$ when $T_1 \longrightarrow +\infty$.*

With this result of stability, we can state precisely the approximation of the solution of (24) by the function u_1.

2.2.3 Convergence of the expansion

Proposition 2.3. *Consider the solution of (24) with*

$$u(0) = \epsilon a_0 + o(\epsilon),\ \ \dot{u}(0) = -\epsilon\omega a_0 \sin(\beta_0) + o(\epsilon),$$

with a_0, β_0 close of the stationary solution $(\bar{a}, \bar{\beta})$,

$$|a_0 - \bar{a}| \leq \epsilon C_1, |\beta_0 - \bar{\beta}| \leq \epsilon C_2.$$

When $\sigma \leq \frac{3d\bar{a}^2}{4\omega} - \frac{1}{2}\sqrt{\frac{3d\bar{a}^2}{2\omega} - \lambda^2}$, there exists $\gamma > 0$ such that for all $t \leq t_\epsilon = \frac{\gamma}{\epsilon}$, the following expansion is satisfied

$$u(t) = \epsilon a(\epsilon t)\ \cos(\tilde{\omega}_\epsilon t + \beta(\epsilon t))\ +\ \epsilon^2 r(\epsilon, t),$$

with $\omega_\epsilon = \omega + \epsilon\sigma$ and r uniformly bounded in $C^2(0, t_\epsilon)$ and with a, β solution of (51).

Proof. Indeed after eliminating terms at frequency 1, we go back to the variable t for the second equation (41)

$$\ddot{r} + \omega^2 r = \frac{\tilde{S}_2}{\omega^2}, \text{ with} \tag{55}$$

$$\tilde{S}_2 = S_2^\sharp(t,\epsilon) - \epsilon \tilde{R}(u_1, r, \epsilon), \tag{56}$$

with

$$\tilde{R}(u_1, r, \epsilon) = R(u_1, r, \epsilon) - \mathcal{D}_2 r - \lambda(\frac{\mathrm{d}r}{\mathrm{d}t} - \omega D_0 r), \tag{57}$$

with all the terms expressed with the variable t; we have

$$S_2^\sharp(t,\epsilon) = \frac{-ca^2(\epsilon t)}{2}(1 + \cos(2(\tilde{\omega}_\epsilon t + \beta(\epsilon t))) - \frac{da^3(\epsilon t)}{4}\left(\cos(3(\tilde{\omega}_\epsilon t + \beta(\epsilon t))\right), \tag{58}$$

this function is not periodic but is *close* to the periodic function:

$$S_2^\natural(t,\epsilon) = \frac{-c\bar{a}^2}{2}(1 + \cos(2(\tilde{\omega}_\epsilon t + \bar{\beta})) - \frac{da^3}{4}\left(\cos(3(\tilde{\omega}_\epsilon t + \bar{\beta})\right) \tag{59}$$

and for $t \leq \frac{\gamma}{\epsilon}$ as the solution of (51) is stable: it remains close to the stationary solution

$$|a(\epsilon t) - \bar{a}| \leq \epsilon C_1, \quad |\beta(\epsilon t) - \bar{\beta}| \leq \epsilon C_2 \tag{60}$$

and

$$|S_2^\sharp - S_2^\natural| \leq \epsilon C_3; \tag{61}$$

so this difference may be included in the remainder $\tilde{R}$. We use lemma 5.1 with $S = S_2^\natural$; it satisfies lemma hypothesis; similarly, we use $g = \tilde{R}$; it satisfies the hypothesis because it is a polynomial in the variables r, u_1, ϵ, with coefficients which are bounded functions, so it is lipschitzian on bounded subsets. □

Remark 2.1. *Under different hypothesis and for systems involving first order derivatives, a result of convergence may be found in [32].*

2.2.4 Maximum of the stationary solution, primary resonance

Consider the stationary solution of (51), it satisfies

$$\left\{\begin{array}{rl} \lambda a\omega & = -F\sin(\beta), \\ a\left(2\omega\sigma - \frac{3da^2}{4}\right) & = -F\cos(\beta), \end{array}\right. \tag{62}$$

manipulating, we get that a is solution of the equation:

$$f(a,\sigma) = \lambda^2 a^2 \omega^2 + a^2\left(2\omega\sigma - \frac{3da^2}{4}\right)^2 - F^2 = 0. \tag{63}$$

We compute

$$\frac{\partial f}{\partial \sigma} = 4a^2\omega(2\omega\sigma - \frac{3da^2}{4}), \tag{64}$$

$$\frac{\partial f}{\partial a} = 2a\lambda^2\omega^2 + 2a\left(2\omega\sigma - \frac{3da^2}{4}\right)^2 - 6\frac{da^3}{4}\left(2\omega\sigma - \frac{3da^2}{4}\right), \tag{65}$$

$$\frac{\partial^2 f}{\partial \sigma^2} = 8a^2\omega^2. \tag{66}$$

$$\tag{67}$$

For σ close enough to the solution of $\frac{\partial f}{\partial \sigma} = 0$, $\frac{\partial f}{\partial \sigma}$ is small, $\frac{\partial f}{\partial a}$ is not zero, and with the implicit function theorem this equation defines a function $a(\sigma)$; lets use :

$$\frac{\partial a}{\partial \sigma} = -\frac{\frac{\partial f}{\partial \sigma}}{\frac{\partial f}{\partial a}} \text{ and } \quad \frac{\partial^2 a}{\partial \sigma^2} = -\frac{\frac{\partial^2 f}{\partial \sigma^2}}{\frac{\partial f}{\partial a}},$$

In our case, when $\frac{\partial a}{\partial \sigma} = 0$, we have

$$\sigma = \frac{3da^2}{8\omega}, \quad \frac{\partial f}{\partial a} = 2a\lambda^2\omega^2, \quad \frac{\partial^2 f}{\partial \sigma^2} = 8a^2\omega^4, \tag{68}$$

so the second derivative $\frac{\partial^2 a}{\partial \sigma^2} < 0$ and a is maximum at the frequency of the free periodic solution.

Proposition 2.4. *The stationary solution of (51) satisfies*

$$\left\{\begin{array}{rl} \lambda a\omega & +F\sin(\beta) = 0, \\ 2a\omega\sigma - \frac{3da^3}{4} & +F\cos(\beta) = 0, \end{array}\right. \tag{69}$$

it reaches its maximum amplitude for $\sigma = \frac{3da^2}{8\omega}$ and $\beta = \frac{\pi}{2} + k\pi$; the excitation is at the angular frequency

$$\tilde{\omega}_\epsilon = \omega + 3\epsilon\frac{da^2}{8\omega} \quad \textit{and} \quad F = \lambda\omega a;$$

it is the angular frequency ν_ϵ of the free periodic solution (23) for this frequency, the approximation (of the solution up to the order ϵ) is periodic:

$$u(t) = \epsilon\frac{F}{\lambda\omega}\sin(\tilde{\omega}_\epsilon t) + \epsilon^2 r(\epsilon, t). \tag{70}$$

Remark 2.2. *We remark that this value of $\sigma = \frac{3da^2}{8\omega}$ is indeed smaller than the maximal value that σ may reach in order that the previous expansion converges as indicated in proposition 2.3.*

Remark 2.3. *We note also that when the stationary solution reaches its maximum amplitude we have $F = \lambda\omega a$ and so we can recover the damping ratio λ from such a forced vibration experiment; this is a close link with the linear case (see for example [33] or the English translation [34]). This is quite interesting in practice as the damping ratio is usually difficult to measure; we have here a kind of stability result for this experiment.*

3 System with strong local cubic non linearity

In the previous section, we have derived a double scale expansion of a solution of a one degree of freedom free vibrations system and damped vibrations with sinusoidal forcing with frequency close to free vibration frequency. Now, we extend the results to the case of multiple degrees of freedom.

3.1 Free vibrations, double scale expansion

We consider a system of vibrating masses attached to springs:

$$M\ddot{u} + Ku + \Phi(u, \epsilon) = 0. \tag{71}$$

The mass matrix M and the rigidity matrix K are assumed to be symmetric and positive definite. We assume that the non linearity is local, all components are zero except for two components $p-1,\ p$ which correspond to the endpoints of some spring assumed to be non linear:

$$\Phi_{p-1}(u, \epsilon) = c(u_p - u_{p-1})^2 + \frac{d}{\epsilon}(u_p - u_{p-1})^3,\ \Phi_p = -\Phi_{p-1}, \quad p = 2, \ldots, n. \tag{72}$$

If the non linear spring would have been the first or the last one, the expression of the function Φ would depend on the boundary condition; each case would be solved using the same method with slight changes in some formulas. In order to get an approximate solution, we are going to write it in the generalized eigenvector basis:

$$K\phi_k = \omega_k M\phi_k,\ \text{with}\ \phi_k^T M\phi_l = \delta_{kl}, \quad k, l = 1 \ldots, n. \tag{73}$$

So we perform the change of function

$$u = \sum_{k=1}^{n} y_k \phi_k, \tag{74}$$

we obtain

$$\ddot{y}_k + \omega_k^2 y_k + \phi_k^T \Phi(\sum_{i=1}^{n} y_i \phi_i, \epsilon) = 0, \quad k = 1 \ldots, n. \tag{75}$$

As Φ has only 2 components which are not zero, it can be written

$$\ddot{y}_k + \omega_k^2 y_k + (\phi_{k,p-1} - \phi_{k,p})\,\Phi_{p-1}(\sum_{i=1}^{n} y_i \phi_i, \epsilon) = 0, \quad k = 1 \ldots, n, \tag{76}$$

or more precisely

$$\ddot{y}_k + \omega_k^2 y_k + (\phi_{k,p-1} - \phi_{k,p}) \left[c \left(\sum_{i=1}^{n} y_i (\phi_{i,p} - \phi_{i,p-1}) \right)^2 + \frac{d}{\epsilon} \left(\sum_{i=1}^{n} y_i (\phi_{i,p} - -\phi_{i,p-1}) \right)^3 \right] = 0, \quad k = 1 \ldots, n. \tag{77}$$

As for the 1 d.o.f. case, we use a double scale expansion to compute an approximate small solution; more precisely, we look for a solution close to the normal mode of the associated linear system; we denote this mode by subscript 1; obviously by permuting the coordinates, this subscript could be anyone (different of p, this case would give similar results with slightly different formulas); we set

$$T_0 = \omega_1 t, \quad T_1 = \epsilon t \tag{78}$$

and we use the *ansatz*:

$$y_k = \epsilon y_k^1(T_0, T_1) + \epsilon^2 r_k(T_0, T_1, \epsilon), \tag{79}$$

so that

$$\frac{\mathrm{d}^2 y_k}{\mathrm{d}t^2} = \epsilon \omega_1^2 D_0^2 y_k^1 + \epsilon^2[2\omega_1 D_0 D_1 y_k^1 + \omega_1^2 D_0^2 r_k] + \epsilon^3[D_1^2 y_k^1 + \mathcal{D}_2 r_k], \tag{80}$$

with

$$\mathcal{D}_2 r_k = \frac{1}{\epsilon}\left(\frac{\mathrm{d}^2 r_k}{\mathrm{d}t^2} - \omega_1^2 D_0^2 r_k\right) = 2\omega_1 D_0 D_1 r_k + \epsilon D_1^2 r_k. \tag{81}$$

We plug previous expansions into (77). By identifying the coefficients of the powers of ϵ in the expansion of (76), we get:

$$\left\{\begin{array}{l} \omega_1^2 D_0^2 y_k^1 + \omega_k^2 y_k^1 = 0, \;\; k = 1\ldots, n, \\ \omega_1^2 D_0^2 r_k + \omega_k^2 r_k = S_{2,k}, \;\; k = 1\ldots, n, \quad \text{with} \end{array}\right. \tag{82}$$

to simplify, the manipulations, we set

$$\delta_p \phi_l = (\phi_{l,p} - \phi_{l,p-1}),$$

so:

$$S_{2,k} \;=\; \frac{-\delta_p\phi_k}{\epsilon^2}\Phi_{p-1}\left(\sum_i (\epsilon y_i^1 + \epsilon^2 r_i)\phi_i, \epsilon\right) \;-\; 2\omega_1 D_0 D_1 y_k^1 \;-\; \epsilon\mathcal{R}_k, \tag{83}$$

with

$$\mathcal{R}_k = \left(D_1^2 y_k^1 + \mathcal{D}_2 r_k\right) \tag{84}$$

and

$$S_{2,k} = \frac{-\delta_p\phi_k}{\epsilon^2}\left[c\left(\sum_i (\epsilon y_i^1 + \epsilon^2 r_i)\delta_p\phi_i\right)^2 + \frac{d}{\epsilon}\left(\sum_i (\epsilon y_i^1 + \epsilon^2 r_i)\delta_p\phi_i\right)^3\right] - 2\omega_1 D_0 D_1 y_k^1 - \epsilon\mathcal{R}_k. \tag{85}$$

The formula may be expanded

$$S_{2,k} = -\delta_p\phi_k\left[c\sum_{i,j} y_i^1 y_j^1 \delta_p\phi_i\delta_p\phi_j + d\sum_{i,j,l} y_i^1 y_j^1 y_l^1 \delta_p\phi_i\delta_p\phi_j\delta_p\phi_l\right] - 2\omega_1 D_0 D_1 y_k^1 - \epsilon R_k\left(y^1, r, \epsilon\right), \tag{86}$$

where

$$R_k(y^1, r, \epsilon) = \mathcal{R}_k + \delta_p\phi_k \left[\epsilon c \sum_{i,j} \left(2y_i^1 r_j + \epsilon r_i r_j\right) \delta_p\phi_i \delta_p\phi_j + \epsilon d \sum_{ijl} \left(3y_i^1 y_j^1 r_l + 3\epsilon y_i^1 r_j r_l + 3\epsilon^2 r_i r_j r_l\right) \delta_p\phi_i \delta_p\phi_j \delta_p\phi_l \right]. \quad (87)$$

We set $\theta(T_0, T_1) = T_0 + \beta(T_1)$ and we note that $D_0\theta = 1,\ D_1\theta = D_1\beta$; we solve the first set of equations (82), imposing $O(\epsilon)$ initial Cauchy data for $k \neq 1$; we get:

$$y_1^1 = a(T_1)\cos(\theta), \quad \text{and } y_k^1 = O(\epsilon),\ k = 2 \dots n; \quad (88)$$

we put terms involving $y_k^1,\ k \geq 2$ into R_k; so we obtain

$$S_{2,1} = -\delta_p\phi_1 \left[c\left(y_1^1 \delta_p\phi_1\right)^2 + d\left(y_1^1 \delta_p\phi_1\right)^3 \right] - 2\omega_1 D_0 D_1 y_1^1 - \epsilon R_1(y^1, r, \epsilon) \text{ and} \quad (89)$$

$$S_{2,k} = -\delta_p\phi_k \left[c\left(y_1^1 \delta_p\phi_1\right)^2 + d\left(y_1^1 \delta_p\phi_1\right)^3 \right] - \epsilon R_k(y^1, r, \epsilon) \text{ for } k \neq 1. \quad (90)$$

Using (88), we get:

$$S_{2,1} = -\delta_p\phi_1 \Big[\frac{ca_1^2}{2}(1 + \cos(2\theta))\left(\delta_p\phi_1\right)^2 + \frac{da_1^3}{4}\left((\cos(3\theta) + 3\cos(\theta))(\delta_p\phi_1)^3\right) \Big] + 2\omega_1(D_1 a_1 \sin(\theta) + a_1 D_1\beta_1 \cos(\theta)) - \epsilon R_1(y^1, r, \epsilon) \text{ and} \quad (91)$$

$$S_{2,k} = -\delta_p\phi_k \Big[\frac{ca_1^2}{2}(1 + \cos(2\theta))\left(\delta_p\phi_1\right)^2 + \frac{da_1^3}{4}\left((\cos(3\theta) + 3\cos(\theta))(\delta_p\phi_1)^3\right) \Big] + - \epsilon R_k(y^1, r, \epsilon) \text{ for } k \neq 1. \quad (92)$$

We gather the terms at angular frequency 1 in $S_{2,1}$:

$$S_{2,1} = -\delta_p\phi_1 \Big[\frac{da_1^3}{4} 3\cos(\theta)(\delta_p\phi_1)^3 \Big] + 2\omega_1(D_1 a_1 \sin(\theta) + a_1 D_1\beta_1 \cos(\theta)) + S_{2,1}^\sharp - \epsilon R(y^1, r, \epsilon), \quad (93)$$

with

$$S^\sharp_{2,1} \quad = \quad -\delta_p\phi_1\left[\frac{ca_1^2}{2}(1+\cos(2\theta))(\delta_p\phi_1)^2+\frac{da_1^3}{4}\cos(3\theta)(\delta_p\phi_1)^3\right]. \quad (94)$$

If we enforce

$$D_1a_1=0, \ \text{ and } \ 2\omega_1 a_1 D_1\beta_1=(\delta_p\phi_1)^4\frac{3da^3}{4}, \text{ so that}$$
$$a_1=a_{1,0}, \quad \beta_1=\beta_{1,0}T_1 \text{ with } \beta_{1,0}=\frac{3da^2}{8\omega}(\delta_p\phi_1)^4T_1, \quad (95)$$

the right hand side

$$S_{2,1}=S^\sharp_{2,1}-\epsilon R_1(y^1,r,\epsilon) \quad (96)$$

contains no term at angular frequency 1; for the other components, without any manipulation, there is no trouble with the frequencies if we assume that all the eigenfrequencies ω_k for $k=2\dots n$ are not multiple of ω_1 ($\omega_k\neq q\omega_1$ for $q=1$ or $q=2$, $q=3$).

In order to prove that r is bounded, after the elimination of terms at frequency 1, we write back the equations with the variable t, for the second set of equations of (82).

$$\omega_1^2\ddot{r}_k+\omega_k^2 r_k=\tilde{S}_{2,k}, \qquad \text{for } k=1,\dots n, \quad (97)$$

with

$$\tilde{S}_{2,1}=S^\sharp_{2,1}-\epsilon\tilde{R}_1(y^1,r,\epsilon), \quad (98)$$

where

$$S^\sharp_{2,1}=-\delta_p\phi_1\left[\frac{ca_1^2}{2}(1+\cos(2(\omega_1t+\beta_{1,0}\epsilon t))(\delta_p\phi_1)^2 \right.$$
$$\left. +\frac{da_1^3}{4}\cos(3(\omega_1t+\beta_{1,0}\epsilon t))(\delta_p\phi_1)^3\right] \quad (99)$$

and

$$\tilde{S}_{2,k}=-\delta_p\phi_k\left[\frac{ca_1^2}{2}(1+\cos(2(\omega_1t+\beta_{1,0}\epsilon t)))\,(\delta_p\phi_1)^2+\right.$$
$$\left.\frac{da_1^3}{4}\left((\cos(3(\omega_1t+\beta_{1,0}\epsilon t))+3\cos((\omega_1t+\beta_{1,0}\epsilon t)))(\delta_p\phi_1)^3\right)\right]$$
$$-\epsilon\tilde{R}_k(y^1,r,\epsilon) \text{ for } k\neq 1 \quad (100)$$

and where

$$\tilde{R}_k(y^1,r,\epsilon)=R_k(y^1,r,\epsilon)-\mathcal{D}_2r_k. \quad (101)$$

Proposition 3.1. *Under the assumption that ω_k and ω_1 are $\mathbb{Z}$ independent for $k \neq 1$, there exists $\gamma > 0$ such that for all $t \leq t_\epsilon = \frac{\gamma}{\epsilon}$, the solution of (76) with initial data*

$$y_1(0) = \epsilon a_{1,0}, \quad \dot{y}_1(0) = 0, \quad y_k(0) = O(\epsilon^2), \quad \dot{y}_k(0) = 0 \tag{102}$$

satisfy the following expansion

$$y_1(t) = \epsilon a_0 \cos(\nu_\epsilon t) + \epsilon^2 r_1(\epsilon, t) \text{ with } \nu_\epsilon = \omega_1 + 3\epsilon \frac{da_0^2}{8\omega_1}(\phi_{1,p} - \phi_{1,p-1})^4, \tag{103}$$

$$y_k(t) = \epsilon^2 r_k(\epsilon, t), \tag{104}$$

with r_k uniformly bounded in $\mathcal{C}^2(0, t_\epsilon)$ for $k = 1, \dots n$ and ω_1, ϕ_1 are the eigenvalue and eigenvectors defined in (73).

Corollary 3.1. *The solution of (71, (72) with*

$$\phi_1^T u(0) = \epsilon a_{1,0}, \quad \phi_1^T \dot{u}(0) = 0, \quad \phi_k^T u(0) = O(\epsilon^2), \quad \phi_k^T \dot{u}(0) = 0,$$

with ω_k, ϕ_k are the eigenvalue and eigenvectors defined in (73)

$$\text{is} \quad u(t) = \sum_{k=1}^{n} y_k(t) \phi_k, \tag{105}$$

with the expansion of y_k of previous proposition.

Proof. For the proposition, we use lemma 5.4. Set $S_1 = S_{2,1}^\sharp, \quad S_k = S_{2,k}$ for $k = 1, \dots n$; as we have enforced (95), the functions S_k are periodic, bounded, and are orthogonal to $e^{\pm it}$, we have assumed that ω_k and ω_1 are $\mathbb{Z}$ independent for $k \neq 1$; so $S_k, \ k = 1, \dots, n$, satisfies the lemma hypothesis. Similarly, set $g = \tilde{R}$, its components are polynomials in r with coefficients which are bounded functions, so it is lipschitzian on the bounded subsets of $\mathbb{R}$, it satisfies the hypothesis of lemma 5.4 and so the proposition is proved. The corollary is an easy consequence of the proposition and the change of function (107) $\square$

Remark 3.1. *We have obtained a periodic asymptotic expansion of a solution of system (71), (72); they are called non linear normal modes in the mechanical community ([23, 22]. In the next section, we shall derive that the frequencies of the normal mode are resonant frequencies for an associated forced system, the so called primary resonance; secondary resonance could be derived along similar lines.*

3.2 Forced, damped vibrations, double scale expansion

3.2.1 Derivation of the expansion

We consider a similar system of forced vibrating masses attached to springs with a light damping:

$$M\ddot{u} + \epsilon C\dot{u} + Ku + \Phi(u, \epsilon) = \epsilon^2 F \cos(\tilde{\omega}_\epsilon t), \tag{106}$$

with the same assumptions as in subsection 3.1. We assume that the frequency of the driving force is close to some frequency of the linearised system (primary resonance); we denote this frequency with the subscript 1: $\tilde{\omega}_\epsilon = \omega_1 + \epsilon\sigma$

We assume that the non linearity is local, all components are zero except for two components $p-1,\ p$ which correspond to the endpoints of some spring assumed to be non linear. As for free vibrations, we perform the change of function

$$u = \sum_{k=1}^{n} y_k \phi_k, \tag{107}$$

with ϕ_k, the generalised eigenvectors of (73). As the damping matrix C is usually not well defined, to simplify, we assume that it is diagonal in the eigenvector basis $\phi_k,\ k = 1,\dots n$. We obtain

$$\ddot{y}_k + \epsilon\lambda_k \dot{y}_k + \omega_k^2 y_k + \phi_k^T \Phi(\sum_{i=1}^{n} y_i\phi_i, \epsilon) = \epsilon^2 f_k \cos(\tilde{\omega}_\epsilon t), \quad k = 1\dots, n, \tag{108}$$

with $f_k = \phi_k^T F$. As for the free vibration case, Φ has only 2 components which are not zero, so the system can be written

$$\ddot{y}_k + \epsilon\lambda_k \dot{y}_k + \omega_k^2 y_k + (\phi_{k,p-1} - \phi_{k,p}, \epsilon)\, \Phi_{p-1}(\sum_{i=1}^{n} y_i\phi_i) = \epsilon^2 f_k \cos(\tilde{\omega}_\epsilon t), \\ k = 1\dots, n, \tag{109}$$

or more precisely

$$\ddot{y}_k + \epsilon\lambda_k \dot{y}_k + \omega_k^2 y_k + (\phi_{k,p-1} - \phi_{k,p}) \left[c \left(\sum_{i=1}^{n} y_i(\phi_{i,p} - \phi_{i,p-1}) \right)^2 + \right. \\ \left. \frac{d}{\epsilon} \left(\sum_{i=1}^{n} y_i(\phi_{i,p} - \phi_{i,p-1}) \right)^3 \right] = \epsilon^2 f_k \cos(\tilde{\omega}_\epsilon t), \\ k = 1\dots, n. \tag{110}$$

As for the 1 d.o.f. case, we use a double scale expansion to compute an approximate small solution; we use a fast scale which is ϵ dependent; we set

$$T_0 = \tilde{\omega}_\epsilon t, \quad T_1 = \epsilon t, \tag{111}$$

and we use the *"ansatz"*

$$y_k = \epsilon y_k^1(T_0, T_1) + \epsilon^2 r_k(T_0, T_1, \epsilon), \tag{112}$$

so that

$$\frac{\mathrm{d}y_k}{\mathrm{d}t} = \epsilon\big[\omega_1 D_0 y_k^1 + \epsilon\sigma D_0 y_k^1 + \epsilon D_1 y_k^1\big] + \epsilon^2\omega_1 D_0 r_k + \epsilon^2(\frac{\mathrm{d}r_k}{\mathrm{d}t} - \omega_1 D_0 r_k), \tag{113}$$

$$\frac{\mathrm{d}^2 y_k}{\mathrm{d}t^2} = \epsilon\Big\{\omega_1^2 D_0^2 y_k^1 + 2\epsilon\omega_1\left[\sigma D_0^2 y_k^1 + D_0 D_1 y_k^1\right] + \epsilon^2\left[\sigma^2 D_0^2 y_k^1 + 2\sigma D_0 D_1 y_k^1 + D_1^2 y_k^1\right]\Big\} + \epsilon^2\omega_1^2 D_0^2 r_k + \epsilon^3 \mathcal{D}_2 r_k, \quad (114)$$

with

$$\mathcal{D}_2 r_k = \frac{1}{\epsilon}\left(\frac{\mathrm{d}^2 r_k}{\mathrm{d}t^2} - \omega_1^2 D_0^2 r_k\right) = 2\omega_1(\sigma D_0^2 r_k + D_0 D_1 r_k) + \epsilon\left[\sigma^2 D_0^2 r_k + 2\sigma D_0 D_1 r_k + D_1^2 r_k\right]. \quad (115)$$

We plug previous expansions into (110). By identifying the coefficients of the powers of ϵ in the expansion of (110), we get:

$$\begin{cases} \omega_1^2 D_0^2 y_k^1 + \omega_k^2 y_k^1 = 0, \ k = 1\dots, n, \\ \omega_1^2 D_0^2 r_k + \omega_k^2 r_k = S_{2,k}, \ k = 1\dots, n, \quad \text{with} \end{cases} \quad (116)$$

$$S_{2,k} = -\left\{\frac{\delta_p\phi_k}{\epsilon^2}\Phi_{p-1}\left(\sum_i(\epsilon y_i^1 + \epsilon^2 r_i)\phi_i, \epsilon\right) + 2\omega_1[D_0 D_1 y_k^1 + \sigma D_0^2 y_k^1] + \lambda_k\omega_1 D_0 y_k^1\right\} + f_k\cos(T_0) - \epsilon R_k(y^1, r, \epsilon), \quad (117)$$

where we gather higher order terms in R_k and to simplify, the manipulations, we have set

$$\delta_p\phi_l = (\phi_{l,p} - \phi_{l,p-1}),$$

so:

$$S_{2,k} = -\frac{\delta_p\phi_k}{\epsilon^2}\left[c\left(\sum_i(\epsilon y_i^1 + \epsilon^2 r_i)\delta_p\phi_i\right)^2 + \frac{d}{\epsilon}\left(\sum_i(\epsilon y_i^1 + \epsilon^2 r_i)\delta_p\phi_i\right)^3\right] - 2\omega_1[D_0 D_1 y_k^1 + \sigma D_0^2 y_k^1] - \lambda_k\omega_1 D_0 y_k^1 + f_k\cos(T_0) - \epsilon R_k(y^1, r, \epsilon). \quad (118)$$

The formula may be expanded

$$S_{2,k} = -\delta_p\phi_k\left[c\sum_{i,j} y_i^1 y_j^1 \delta_p\phi_i\delta_p\phi_j + d\sum_{i,j,l} y_i^1 y_j^1 y_l^1 \delta_p\phi_i\delta_p\phi_j\delta_p\phi_l\right] - 2\omega_1[D_0 D_1 y_k^1 + \sigma D_0^2 y_k^1] - \lambda_k\omega_1 D_0 y_k^1 + f_k\cos(T_0) - \epsilon R_k(y^1, r, \epsilon). \quad (119)$$

We set $\theta(T_0, T_1) = T_0 + \beta(T_1)$ and we note that $D_0\theta = 1,\ D_1\theta = D_1\beta$; we solve the first set of equations (116), imposing initial Cauchy data for $k \neq 1$ of order $O(\epsilon)$ we get:

$$y_1^1 = a_1(T_1)\cos(\theta), \quad \text{and } y_k^1 = O(\epsilon),\ k = 2\dots n; \tag{120}$$

we put terms involving y_k^1 into R_k for $k \geq 2$ and so we obtain

$$\begin{aligned} S_{2,1} &= -\delta_p\phi_1 \left[c\left(y_1^1\delta_p\phi_1\right)^2 + d\left(y_1^1\delta_p\phi_1\right)^3\right] \\ &- 2\omega_1[D_0D_1y_1^1 + \sigma D_0^2 y_1^1] - \lambda_1\omega_1 D_0 y_1^1 + f_1\cos(T_0) - \epsilon R_1(y^1, r, \epsilon) \text{ and} \end{aligned} \tag{121}$$

$$\begin{aligned} S_{2,k} &= -\delta_p\phi_k \left[c\left(y_1^1\delta_p\phi_1\right)^2 + d\left(y_1^1\delta_p\phi_1\right)^3\right] + \\ &\qquad f_k\cos(T_0) - \epsilon R_k(y^1, r, \epsilon) \text{ for } k \neq 1. \end{aligned} \tag{122}$$

Using (120), we get:

$$\begin{aligned} S_{2,1} = -\delta_p\phi_1 \Big[&\frac{ca_1^2}{2}(1+\cos(2\theta))\left(\delta_p\phi_1\right)^2 + \\ &\frac{da_1^3}{4}\left((\cos(3\theta) + 3\cos(\theta))(\delta_p\phi_1)^3\right)\Big] + \\ &2\omega_1[D_1a_1\sin(\theta) + a_1D_1\beta_1\cos(\theta) + \sigma a_1\cos(\theta)] + \lambda_1\omega_1 a_1\sin(\theta) \\ &+ f_1[\sin(\theta)\sin(\beta) + \cos(\theta)\cos(\beta)] - \epsilon R_1(y^1, r, \epsilon) \text{ and} \end{aligned} \tag{123}$$

$$\begin{aligned} S_{2,k} = -\delta_p\phi_k \Big[&\frac{ca_1^2}{2}(1+\cos(2\theta))\left(\delta_p\phi_1\right)^2 + \\ &\frac{da_1^3}{4}\left((\cos(3\theta) + 3\cos(\theta))(\delta_p\phi_1)^3\right)\Big] + \\ &+ f_k[\sin(\theta)\sin(\beta) + \cos(\theta)\cos(\beta)] - \epsilon R_k(y^1, r, \epsilon) \text{ for } k \neq 1. \end{aligned} \tag{124}$$

We gather the terms at angular frequency 1 in $S_{2,1}$

$$\begin{aligned} S_{2,1} = \delta_p\phi_1 &\left[-3\frac{da_1^3}{4}\cos(\theta)(\delta_p\phi_1)^3 + 2\omega_1(a_1D_1\beta_1 + \sigma a_1) + f_1\cos(\beta)\right]\cos(\theta) \\ &+ \Big[\omega_1(2D_1a_1 + \lambda_1 a_1) + f_1\sin(\beta)\Big]\sin(\theta) + S_{2,1}^\sharp - \epsilon R(y^1, r, \epsilon), \end{aligned} \tag{125}$$

with

$$S_{2,1}^\sharp \quad = \quad -\delta_p\phi_1\left[\frac{ca_1^2}{2}(1+\cos(2\theta))(\delta_p\phi_1)^2 + \frac{da_1^3}{4}\cos(3\theta)(\delta_p\phi_1)^3\right]. \tag{126}$$

Orientation If we enforce

$$\left\{\begin{array}{rl} \omega_1\big(2D_1a_1+\lambda_1a_1\big) & = -f_1\sin(\beta_1), \quad \text{and} \\ 2\omega_1\big(a_1D_1\beta_1+\sigma a_1\big) & = \frac{3da^3}{4}(\delta_p\phi_1)^4 - f_1\cos(\beta_1), \end{array}\right. \tag{127}$$

the right hand side

$$S_{2,1} = S^\sharp_{2,1} - \epsilon R_1(y^1, r, \epsilon) \tag{128}$$

contains no term at angular frequency 1; for the other components, without any manipulation, there is not such terms , if we assume that all the eigenfrequencies ω_k for $k = 2\ldots n$ are not multiple of ω_1 ($\omega_k \neq q\omega_1$ for $q = 1$ or $q = 2$, $q = 3$). This will enable us to justify this expansion; previously, we study the stationary solution of this approximate system and the stability of the solution in a neighbourhood of this stationary solution.

3.2.2 Stationary solution and stability

The situation is very close to the 1 d.o.f. case; except the replacement of d by of $\tilde{d} = d(\delta_p\phi_1)^4$, the system (127) is the same as (51); the other components are zero. We state a similar proposition

Proposition 3.2. *When $\sigma \leq \frac{3\tilde{d}\bar{a}^2}{4\omega} - \frac{1}{2}\sqrt{\frac{9\tilde{d}^2\bar{a}^4}{16\omega^2} - \lambda_1^2}$, the stationary solution of (127) is stable in the sense of Lyapunov (if the dynamic solution starts close to the stationary one, it remains close and converges to it); to the stationary case corresponds the approximate solution of (77 $y^1_1 = \bar{a}_1\cos(T_0+\bar{\beta}_1)$, $\quad y^1_k = O(\epsilon)$, $\quad k = 2,\ldots,n$, it is periodic; for an initial data close enough to the stationary solution, $y^1_1 = a(T_1)\cos(T_0+\beta_1(T_1))$, $\quad y^1_k = O(\epsilon)$, $\quad k = 2,\ldots,n$ with a, β_1 solutions of (127) with d replaced by $\tilde{d}$; they converge to the stationary solution $\bar{a}_1, \bar{\beta}_1$ when $T_1 \longrightarrow +\infty$.*

3.2.3 Convergence of the expansion

In order to prove that r is bounded, after the elimination of terms at frequency 1, we write back the equations with the variable t, for the second set of equations of (82).

$$\omega_1^2\ddot{r}_k + \omega_k^2 r_k = \tilde{S}_{2,k} \quad \text{for } k = 1,\ldots n, \tag{129}$$

with

$$\tilde{S}_{2,1} = S^\sharp_{2,1} - \epsilon\tilde{R}_1(y^1, r, \epsilon), \tag{130}$$

where

$$S^\sharp_{2,1} = -\delta_p\phi_1\left[\frac{c(a_1(\epsilon t))^2}{2}(1+\cos(2(\tilde{\omega}_\epsilon t+\beta_1(\epsilon t))(\delta_p\phi_1)^2 \right. \\ \left. + \frac{da_1^3}{4}\cos(3(\tilde{\omega}_\epsilon t+\beta_1(\epsilon t))(\delta_p\phi_1)^3\right] \tag{131}$$

and

$$S_{2,k} = -\delta_p\phi_k\Bigg[\frac{c(a_1(\epsilon t)^2}{2}(1+\cos(2(\tilde{\omega}_\epsilon t+\beta_1(\epsilon t)))\,(\delta_p\phi_1)^2 + \\ \frac{da_1^3}{4}\left((\cos(3(\tilde{\omega}_\epsilon t+\beta_(\epsilon t))+3\cos((\tilde{\omega}_\epsilon t+\beta_1(\epsilon t)))(\delta_p\phi_1)^3\right)\Bigg] \\ -\epsilon R_k(y^1,r,\epsilon) \text{ for } k\neq 1, \quad (132)$$

where

$$\tilde{R}_k(y^1,r,\epsilon) = R_k(y^1,r,\epsilon) - \mathcal{D}_2 r_k - \lambda_k\big(\frac{\mathrm{d}r_k}{\mathrm{d}t} - \omega_k D_0 r_k\big). \quad (133)$$

Proposition 3.3. *Under the assumption that ω_k and ω_1 are $\mathbb{Z}$ independent for $k\neq 1$, there exists $\gamma>0$ such that for all $t\leq t_\epsilon = \frac{\gamma}{\epsilon}$, the solution of (110) with initial data*

$$y_1(0) = \epsilon a_{1,0} + O(\epsilon^2), \quad \dot{y}_1(0) = -\epsilon\omega a_{1,0}\sin(\beta_{1,0}) + O(\epsilon^2), \quad (134)$$

$$y_k(0) = O(\epsilon^2), \quad \dot{y}_k(0) = 0 \quad (135)$$

and with the initial data close to the stationary solution

$$|a_{1,0}-\bar{a}_1| \leq \epsilon C_1, \quad |\beta_{1,0}-\bar{\beta}_1|, \leq \epsilon C_1$$

satisfy the following expansion

$$y_1(t) = \epsilon a_1(\epsilon t)\cos(\tilde{\omega}_\epsilon t + \beta_1(\epsilon t)) + \epsilon^2 r_1(\epsilon,t), \text{ with} \quad (136)$$

$$y_k(t) = \epsilon^2 r_k(\epsilon,t), \quad (137)$$

with a_1,β_1 solution of (127) and with r_k uniformly bounded in $\mathcal{C}^2(0,t_\epsilon)$ for $k=1,\dots n$ and ω_1,ϕ_1 are the eigenvalue and eigenvectors defined in (73 and $a_1\beta_1$ are solution of (127)

Corollary 3.2. *The solution of (106), (72) with*

$$\phi_1^T u(0) = \epsilon a_{1,0}, \quad \phi_1^T \dot{u}(0) = -\epsilon\omega_1 a_{1,0}\sin(\beta_{1,0}), \quad \phi_k^T u(0) = O(\epsilon^2), \quad \phi_k^T\dot{u}(0) = 0,$$

with ω_k,ϕ_k the eigenvalues and eigenvectors defined in (73),

$$\textit{is} \quad u(t) = \sum_{k=1}^n y_k(t)\phi_k, \quad (138)$$

with the expansion of y_k of previous proposition.

Proof. For the proposition, we use lemma 5.4. Set $S_1 = S_{2,1}^\sharp$, $S_k = S_{2,k}$ for $k = 1,\dots n$; as we have enforced (95), the functions S_k are periodic, bounded, and are orthogonal to $e^{\pm it}$, we have assumed that ω_k and ω_1 are $\mathbb{Z}$ independent for $k\neq 1$; so S satisfies the lemma hypothesis. Similarly, set $g=\tilde{R}$, it is a polynomial in r with coefficients which are bounded functions , so it is lipschitzian on the bounded subsets of $\mathbb{R}$, it satisfies the hypothesis of lemma 5.4 and so the proposition is proved. The corollary is an easy consequence of the proposition and the change of function (107) ☐

3.2.4 Maximum of the stationary solution

As equation (127) is similar to the equation (51) of the 1 d.o.f. case, we get also that the stationary solution reaches its maximum amplitude to the frequency of the free periodic solution.

Consider the stationary solution of (127), it satisfies

$$\left\{ \begin{array}{rl} \lambda_1 a_1 \omega_1 & = -f_1 \sin(\beta_1), \\ a \left(2\omega_1 \sigma - \frac{3\tilde{d}a^2}{4} \right) & = -f_1 \cos(\beta_1); \end{array} \right. \tag{139}$$

manipulating, we get that a_1 is solution of the equation:

$$f(a_1, \sigma) = \lambda_1^2 a_1^2 \omega^2 + a_1^2 \left(2\omega_1 \sigma - \frac{3\tilde{d}a_1^2}{4} \right)^2 - f_1^2 = 0. \tag{140}$$

As for the 1 d.o.f. case, we can state:

Proposition 3.4. *The stationary solution of (127) satisfies*

$$\left\{ \begin{array}{rl} \lambda_1 a_1 \omega_1 & + f_1 \sin(\beta_1) = 0, \\ 2a_1 \omega_1 \sigma - \frac{3\tilde{d}a^3}{4} & + f_1 \cos(\beta_1) = 0, \end{array} \right. \tag{141}$$

it reaches its maximum amplitude for $\sigma = \frac{3\tilde{d}a_1^2}{8\omega_1}$ and $\beta_1 = \frac{\pi}{2} + k\pi$; the excitation is at the frequency

$$\tilde{\omega}_\epsilon = \omega_1 + 3\epsilon \frac{\tilde{d}a_1^2}{8\omega_1}, \quad \textit{with} \quad \tilde{d} = d(\Phi_{1,p} - \Phi_{1,p-1})^4 \quad \textit{and} \quad F = \lambda_1 \omega_1 a_1$$

where Φ_1 is the eigenvector of the underlying linear system associated to ω_1; $\tilde{\omega}_\epsilon$ is the frequency of the free periodic solution (23); for this frequency, the approximation (of the solution up to the order ϵ) is periodic:

$$y_1(t) = \epsilon \frac{f_1}{\lambda_1 \omega_1} \sin(\tilde{\omega}_\epsilon t) + \epsilon^2 r(\epsilon, t), \tag{142}$$

$$y_k(t) = \epsilon^2 r_k(\epsilon, t). \tag{143}$$

As for the 1 d.o.f. case we can remark the following points.

Remark 3.2. *This value of $\sigma = \frac{3\tilde{d}a_1^2}{8\omega_1}$ is indeed smaller than the maximal value that σ may reach in order that the system be stable and that the previous expansion converges as indicated in proposition 2.3.*

Remark 3.3. *We note also that when the stationary solution reaches its maximum amplitude we have $f_1 = \lambda_1 \omega_1 a_1$ and so we can recover the damping ratio λ_1 from such a forced vibration experiment; this is a close link with the linear case (see for example [33] or the English translation [34]). This is quite interesting in practice as the damping ratio is usually difficult to measure. Obviously, we can recover the damping ratio for other frequencies by performing other experiments.*

We can also consider this result as a stability of the process used in the linear case with respect to the appearance of a small non-linearity.

4 Conclusion

For differential systems modeling spring-masses vibrations with non linear springs, we have derived and rigorously proved an asymptotic approximation of periodic solution of free vibrations (so called non linear normal modes); for damped vibrations with periodic forcing with frequency close to free vibration frequency (the so called primary resonance case), we have obtained an asymptotic expansion and derived that the amplitude is maximal at the frequency of the non linear normal mode.

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5 Appendix

Lemma 5.1. *Let w_ϵ be solution of*

$$w" + w = S(t, \epsilon) + \epsilon g(t, w, \epsilon), \tag{144}$$

$$w(0) = 0, \quad w'(0) = 0. \tag{145}$$

If the right hand side satisfies the following conditions

1. *S is a sum of periodic bounded functions:*
   (a) *for all t and for all ϵ small enough, $S(t, \epsilon) \leq M$,*
   (b) *$\int_0^{2\pi} e^{it} S(t, \epsilon) dt = 0$, $\int_0^{2\pi} e^{-it} S(t, \epsilon) dt = 0$ uniformly for ϵ small enough*
2. *for all $R > 0$, there exists k_R such that for $|u| \leq R$ and $|v| \leq R$, the inequality $|g(t, u, \epsilon) - g(t, v, \epsilon)| \leq k_R |u - v|$ holds and $|g(t, 0, \epsilon)|$is bounded; in other words g is locally lipschitzian with respect to u.*

then, there exists $\gamma > 0$ such that for ϵ small enough, w_ϵ is uniformly bounded in $C^2(0, T_\epsilon)$ with $T_\epsilon = \frac{\gamma}{\epsilon}$

Proof. The proof is close to the proof of lemma 6.3 of [15]; but it is technically simpler since here we assume g to be locally lipschitzian with respect to u whereas it is only bounded in [15].

1. We first consider

$$w_1" + w_1 = S(t, \epsilon), \tag{146}$$

$$w_1(0) = 0, \quad w_1'(0) = 0; \tag{147}$$

   as S is a sum of periodic functions which are uniformly orthogonal to e^{it} and e^{-it}, w_1 is bounded in $\mathcal{C}^2(0, +\infty)$.

2. Then we perform a change of function: $w = w_1 + w_2$, the following equalities hold

$$w_2" + w_2 = \epsilon g_2(t, w_2, \epsilon), \tag{148}$$

$$w_2(0) = 0, \quad w_2'(0) = 0, \tag{149}$$

with g_2 which satisfies the same hypothesis as g:

for all $R > 0$, there exists k_R such that for $|u| \leq R$ and $|v| \leq R$, the following inequality holds $|g_2(t, u, \epsilon) - g_2(t, v, \epsilon)| \leq k_R|u - v|$. Using Duhamel principle, the solution of this equation satisfies:

$$w_2 = \epsilon \int_0^t \sin(t-s) g_2(s, w_2(s), \epsilon) \mathrm{d}s \tag{150}$$

from which

$$|w_2(t)| \leq \epsilon \int_0^t |g_2(s, w_2(s), \epsilon) - g_2(s, 0, \epsilon)| \mathrm{d}s + \epsilon \int_0^t |g_2(s, 0, \epsilon)| \mathrm{d}s \tag{151}$$

so if $|w| \leq R$, hypothesis of lemma imply

$$|w_2(t)| \leq \epsilon \int_0^t k_R |w_2| \mathrm{d}s + \epsilon C t. \tag{152}$$

A corollary of lemma of Bellman-Gronwall, see below, will enable to conclude. It yields

$$|w_2(t)| \leq \frac{C}{k_R} \left(\exp(\epsilon k_R t) - 1\right). \tag{153}$$

Now set $T_\epsilon = \sup\{t \| w| \leq R\}$, then we have

$$R \leq \frac{C}{k_R} \left(\exp(\epsilon k_R t) - 1\right);$$

this shows that there exists γ such that $|w_2| \leq R$ for $t \leq T_\epsilon$, which means that it is in $L^\infty(0, T_\epsilon)$ for $T_\epsilon = \frac{\gamma}{\epsilon}$; also, we have w in $\mathcal{C}(0, T_\epsilon)$ then as w is solution of (144), it is also bounded in $\mathcal{C}^2(0, T_\epsilon)$.

□

Lemma 5.2. *(Bellman-Gronwall, [35, 36]) Let u, ϵ, β be continuous functions with $\beta \geq 0$,*

$$u(t) \leq \epsilon(t) + \int_0^t \beta(s) u(s) \mathrm{d}s \textit{ for } 0 \leq t \leq T, \tag{154}$$

then

$$u(t) \leq \epsilon(t) + \int_0^t \beta(s) \epsilon(s) \left[\exp(\int_s^t \beta(\tau) d\tau\right] \mathrm{d}s. \tag{155}$$

Lemma 5.3. *(a consequence of previous lemma, suited for expansions, see [12]) Let u be a positive function, $\delta_2 \geq 0$, $\delta_1 > 0$ and*

$$u(t) \leq \delta_2 t + \delta_1 \int_0^t u(s)\mathrm{d}s,$$

then

$$u(t) \leq \frac{\delta_2}{\delta_1}\left(\exp(\delta_1 t) - 1.\right)$$

Lemma 5.4. *Let $v_\epsilon = [v_1^\epsilon, \ldots, v_N^\epsilon]^T$ be the solution of the following system:*

$$\omega_1^2 (v_k^\epsilon)'' + \omega_k^2 v_k^\epsilon = S_k(t) + \epsilon g_k(t, v_\epsilon). \tag{156}$$

If ω_1 and ω_k are $\mathbb{Z}$ independent for all $k = 2 \ldots N$ and the right hand side satisfies the following conditions with $M > 0$, $C > 0$ prescribed constants:

1. *S_k is a sum of bounded periodic functions, $|S_k(t)| \leq M$ which satisfy the non resonance conditions:*
2. *S_1 is orthogonal to $e^{\pm it}$, i.e. $\int_0^{2\pi} S_1(t)e^{\pm it}dt = 0$ uniformly for ϵ going to zero;*
3. *for all $R > 0$ there exists k_R such that for $\|u\| \leq R$, $\|v\| \leq R$, the following inequality holds for $k = 1, \ldots, N$:*

$$|g_k(t, u, \epsilon) - g_k(t, v, \epsilon)| \leq k_R \|u - v\|$$

and $|g_k(t, 0, \epsilon)|$ is bounded

then there exists $\gamma > 0$ such that for ϵ small enough v_ϵ is bounded in $\mathcal{C}^2(0, T_\epsilon)$ with $T_\epsilon = \frac{\gamma}{\epsilon}$

Proof. 1. We first consider the linear system

$$\omega_1^2 (v_{k,1})'' + \omega_k^2 v_{k,1} = S_k, \tag{157}$$

$$v_{k,1}(0) = 0 \text{ and } (v_{k,1})' = 0. \tag{158}$$

For $k = 1$, with hypothesis 1.a, S_1 is a sum of bounded periodic functions; it is orthogonal to $e^{\pm it}$, there is no resonance. For $k \neq 1$, there is no resonance as $\frac{\omega_k}{\omega_1} \notin \mathbb{Z}$ with hypothesis 1.b.

So $v_{k,1}$ belongs to $\mathcal{C}^{(\in)}(\prime, \mathcal{T}_\epsilon)$ for $k = 1, ..., n$.

2. Then we perform a change of function

$$v_k^\epsilon = v_{k,1} + v_{k,2}^\epsilon$$

and $v_{k,2}^\epsilon$ are solutions of the following system :

$$\omega_1^2(v_{k,2})\text{''} + \omega_k^2 v_{k,2} = \epsilon g_{k,2}(t, v_{k,2}, \epsilon),\ k = 1, \ldots, N, \tag{159}$$

$$v_{k,2}^\epsilon(0) = 0,\ (v_{k,2}^\epsilon)' = 0,\ k = 1, \ldots, N, \tag{160}$$

with

$$g_{k,2}(t, ..., v_{k,2}^\epsilon, ...) = g_k(t, ..., v_{k,1} + v_{k,2}^\epsilon, ...),$$

where $g_{k,2}$ satisfies the same hypothesis as g_k:
for all $R > 0$ there exists k_R such that for $\| u_k \| \leq R$, $\| v_k \| \leq R$, the following inequality holds for $k = 1, \ldots, N$:

$$\| g_{k,2}(t, u_k, \epsilon) - g_{k,2}(t, v_k, \epsilon) \| \leq k_R \| u_k - v_k \| . \tag{161}$$

Using Duhamel principle, the solution or the equation (159) satisfies:

$$v_{k,2}^\epsilon = \epsilon \int_0^t \sin(t-s) g_{k,2}(s, v_{k,2}^\epsilon(s), \epsilon) \mathrm{d}s, \tag{162}$$

so

$$\| v_{k,2}^\epsilon(t) \| \leq \epsilon \int_0^t \| g_{k,2}(s, v_{k,2}^\epsilon(s), \epsilon) - g_{k,2}(s, 0, \epsilon) \| \mathrm{d}s + \epsilon \int_0^t \| g_{k,2}(s, 0, \epsilon) \| \mathrm{d}s, \tag{163}$$

so with (161), we obtain

$$\| v_{k,2}^\epsilon(t) \| \leq \epsilon \int_0^t k \| v_{k,2}^\epsilon(t) \| \mathrm{d}s + \epsilon C t. \tag{164}$$

We shall conclude using Bellman-Gronwall lemma; we obtain

$$\| v_{k,2}(t) \| \leq \frac{C}{k_R}(\exp(\epsilon k_R t) - 1); \tag{165}$$

this shows that there exists γ such that $|v_{k,2}^\epsilon| \leq R$ for $t \leq T_\epsilon$, which means that it is in $L^\infty(0, T_\epsilon)$ for $T_\epsilon = \frac{\gamma}{\epsilon}$; also, we have v_k in $\mathcal{C}(0, T_\epsilon)$ then as v_k is solution of (144), it is also bounded in $\mathcal{C}^2(0, T_\epsilon)$.

□

Theorem 5.1. *(of Poincaré-Lyapunov, for example see [12]) Consider the equation*

$$\dot{x} = (A + B(t))x + g(t, x),\ x(t_0) = x_0,\ t \geq t_0$$

where $x, x_0 \in \mathbf{R}^n$, A is a constant matrix $n \times n$ with all its eigenvalues with negative real parts; $B(t)$ is a matrix which is continuous with the property $\lim_{t \to +\infty} \|B(t)\| = 0$. The vector field is continuous with respect to t and x

is continuously differentiable with respect to x *in a neighborhood of* $x = 0$*; moreover*

$$g(t,x) = o(\|x\|) \text{ when } \|x\| \to 0$$

uniformly in t*. Then, there exists constants* C, t_0, δ, μ *such that if*

$$\|x_0\| < \frac{\delta}{C}$$

$$\|x\| \le C\|x_0\|e^{-\mu(t-t_0)}, t \ge t_0$$

holds

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